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Laminar boundary-layer flow past a two-dimensional slot

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Observations by Cornhill & Roach (1976) of sudanophilic lesions in the vicinity of intercostal arteries in rabbit aortas have shown that lesions develop on the downstream side of the associated ostia. There is considerable conjecture as to the role which varying levels of wall shear stress play in the development of such lesions; Cornhill & Roach implicate high wall shear stress levels. We develop a consistent model of steady boundary-layer flow past a side slot assuming that there is Stokes flow in the side slot and that the main body of the boundary layer remains undisturbed. Our results show that increased levels of wall shear stress occur both upstream and downstream of the slot. If the withdrawal of fluid through the side slot is sufficiently great there may be a stagnation point on the downstream side of the slot. The wall shear stress level near the slot depends on both normal and transverse motions at the mouth of the slot. Indeed, very near the slot, on a length scale comparable with the slot width, the wall shear stress level depends only on the transverse motions at the mouth of the slot.

1. Introduction

Recently Cornhill & Roach (1976), hereafter referred to as CR, have reported a quantitative study of the localization of atherosclerotic lesions in the rabbit aorta. The motivation for the present work arose out of observations reported to us at a preliminary stage by Roach (1975, private communication). In their study CR measured the size and location of early sudanophilic lesions in the aortas of five rabbits. They found that a lesion was most likely to be observed near an orifice. Generally the lesion would be located downstream of the associated ostia, the entrance to the intercostal artery. This observation has prompted CR to the conclusion that their work gives further substantiation to Fry's hypothesis that increased levels of wall shear stress increase the permeability of the arterial wall to lipids in the bloodstream (see Fry 1973). Underlying the conclusions of CR are mathematical (Davids & Kandarpa 1974) and experimental models (Lutz, Cannon & Munroe 1974) of flow in symmetric bifurcations, which predict that, in the area of the flow divider in a bifurcation, elevated wall shear stress levels are to be expected. However since the situation near the intercostal ostia differs substantially from that of a symmetric bifurcation we feel that further evidence is necessary to support the conclusions of CR.

Briefly, the intercostal arteries are small vessels which carry blood to the intercostal muscles, muscles attached to the rib cage which are available (but not always used) to aid breathing. Only a small fraction of the blood flowing down the aorta actually passes into the intercostals. In the rabbits treated by CR areas of intercostal ostia

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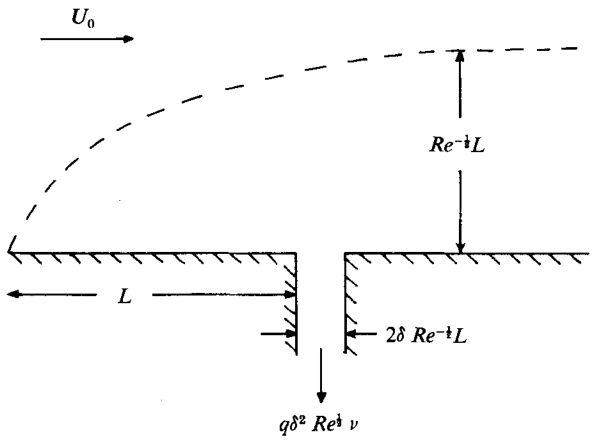

FIGURE 1. Laminar boundary-layer flow past a side slot.

were of the order of 0·3–0·5 mm², giving a radius of about 0·3 mm. The descending aorta would be of radius 2–3 mm.

It would seem that the model closest to the physiological situation is one of a side tube joining a much larger tube with a small withdrawal fraction through the side tube. Recently Smith (1976) has modelled injection into a cylindrical tube using the techniques of modern boundary-layer theory. In order to make his model tractable Smith assumed that the flow at the entrance to the side tube was normal to the wall of the main tube. Using this assumption he derived a solution in the main tube by means of a triple-deck analysis. We show below that his solution is unlikely to be accurate very near the side tube, whilst it is a good approximation far from the side tube.

In the model we present here an important feature is the existence of transverse velocities at the mouth of the side tube. We assume that the side tube adjoins a semi-infinite region of boundary-layer flow over a plate. By making this assumption we are to a certain extent invalidating the application of our theory to flow near the intercostals, but because of the small ratio of the intercostal radius to the aortic radius, we should still expect that the flow over a flat plate would be a good approximation to the actual flow. Thus whilst we choose to ignore the global geometry and concentrate on the local dynamics, Smith has ignored the local dynamics to obtain a model of the flow in the main tube far from the side tube. It is to be hoped that the two theories will result in an adequate picture of flow in an asymmetric bifurcation.

Consider a flat plate which has a side slot joined to it at a distance L from the leading edge (see figure 1). Over the plate there is a laminar boundary-layer flow with free-stream velocity U_0. The fluid has kinematic viscosity ν and we define a Reynolds number $Re = U_0 L/\nu$. Suppose that the side slot has a half-width

$$a = \delta\, Re^{-\frac{1}{2}} L,$$

which is small compared with the boundary-layer thickness near the slot ($Re^{-\frac{1}{2}}L$), so that $\delta \ll 1$. Let the flux through the slot, in either direction, be $q\delta^2\, Re^{\frac{1}{2}}\nu$. Near the slot, and in the slot, we define a region called the basement (see figure 2), in which the equations of motion when scaled to the dimensions of the side slot are to leading order

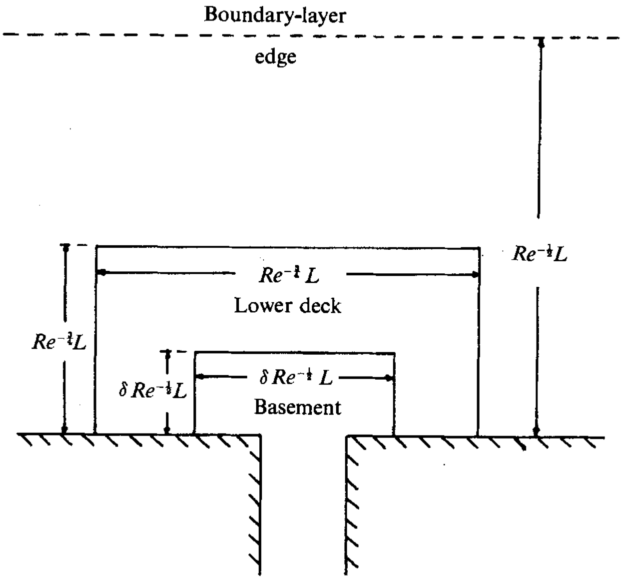

FIGURE 2. Structure of solution for flow through a side slot.

in $\delta Re^{\frac{1}{2}}$ the Stokes equations of motion. In the basement it is possible to find an analytic solution for the regions above and below the line of the mouth of the slot. However we find these solutions impractical and compute the solution to the two-dimensional biharmonic equation using a finite-difference method. A point that arises from the analytic solution is that in the basement region the shear stress on the wall of the plate is dependent only on the transverse velocities at the mouth of the slot and on the mainstream shear. In view of the conventional assumption that transverse velocities at the mouth of the slot vanish, we feel this observation to be worthy of wide attention.

Far from the slot, on a length scale of $Re^{-\frac{3}{4}}L$, there is a region analogous to the lower deck of a triple-deck analysis, with the mathematical complication (but physical simplification) that the length scales in the directions normal and transverse to the plate are identical. Such a region has been considered before by Stewartson (1968) and Smith (1973). The analysis of the lower-deck region shows that, far from the side slot and on the plate, the shear stress decays like $|\hat{x}|^{-\frac{4}{3}}$ and depends on both the normal and the tangential motions at the mouth of the slot. However, on the plate with

$$\delta Re^{-\frac{1}{2}}L \ll |\hat{x}| \ll Re^{-\frac{3}{4}}L$$

the behaviour of the shear is like $|\hat{x}|^{-2}$ and to leading order is independent of motions normal to the mouth of the slot, thus matching with the result obtained for the basement region.

If one goes further from the side slot the disturbance caused by the slot is smaller than the perturbation caused by the boundary layer growing over the plate, and essentially in our analysis there is no disturbance to the main body of the boundary layer. Our solution to the two-dimensional problem can give only broad guidelines to physiological problems and we are attempting to establish a solution to the corresponding three-dimensional problem.

2. Analysis of the basement region

On the length scale of the side slot the main body of the boundary layer is far away and locally there is linear shear flow past the slot. As the flux through the hole is assumed small the transverse and normal length scales are identical, and to leading order in $\delta\, Re^{-\frac{1}{4}}$ the local flow is Stokes-like.

Define $\xi = \hat{x}/a$ and $\eta = \hat{y}/a$ and if $\eta \geqslant 0$ let

$$\hat{u} = U_0[\mathscr{U}(\delta\eta) + q\delta\, \partial\psi_+/\partial\eta] \tag{1a}$$

and

$$\hat{v} = -U_0 q\delta\, \partial\psi_+/\partial\xi. \tag{1b}$$

Then the equations motion reduce to a single equation for the stream function ψ_+:

$$\nabla^4\psi_+ + \frac{\delta^2}{q}\mathscr{U}'''(\delta\eta) = \delta\, Re^{\frac{1}{4}}\{\mathscr{U}(\delta\eta)\nabla^2\psi_{+\xi} - \delta^2\mathscr{U}''(\delta\eta)\,\psi_{+\xi}\} + q\delta^2\, Re^{\frac{1}{4}}\,\frac{\partial(\nabla^2\psi_+, \psi_+)}{\partial(\xi,\eta)}. \tag{2}$$

When $\eta < 0$ we take

$$\hat{u} = U_0 q\delta\partial\psi_-/\partial\eta \tag{3a}$$

and

$$\hat{v} = -U_0 q\delta[\tfrac{3}{4}(1-\xi^2) + \partial\psi_-/\partial\xi]. \tag{3b}$$

The equation for the stream function ψ_- is then

$$\nabla^4\psi_- = q\delta^2\, Re^{\frac{1}{4}}\{\partial(\nabla^2\psi_-, \psi_-)/\partial(\xi,\eta) + \tfrac{3}{4}(1-\xi^2)\nabla^2\psi_{-\eta} - \tfrac{3}{2}\psi_{-\eta}\}. \tag{4}$$

The scaling (1) and (3) of the velocities breaks down when $|q| = 0$ and as $|q| \rightarrow \infty$. For the present we exclude these cases, assuming $|q| = O(1)$. The matching conditions between ψ_+ and ψ_- are that the shear and normal stress should be continuous on $\eta = 0$, for $|\xi| < 1$, as should the velocities.

To match the velocities across $\eta = 0$,

$$[\partial\psi_+/\partial\eta]_{\eta=0^+} = [\partial\psi_-/\partial\eta]_{\eta=0^-}, \quad |\xi| < 1, \tag{5a}$$

and

$$[\partial\psi_+/\partial\xi]_{\eta=0^+} = [\partial\psi_-/\partial\xi]_{\eta=0^-} + \tfrac{3}{4}(1-\xi^2), \quad |\xi| < 1. \tag{5b}$$

The conditions for continuity of stress are, if $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{n} = (0,1)$ the normal to the surface,

$$(\boldsymbol{\sigma}|_{\eta=0^+}) \cdot \mathbf{n} = (\boldsymbol{\sigma}|_{\eta=0^-}) \cdot \mathbf{n}. \tag{6}$$

Using a plus or a minus subscript to represent values of a function for $\eta \geqslant 0$ and $\eta < 0$ respectively, the stress continuity conditions reduce to

$$p_+ = p_- \quad \text{on } \eta = 0, \quad |\xi| < 1, \tag{7}$$

and

$$\mathscr{U}'(0) + q\psi_{+\eta\eta} = q\psi_{-\eta\eta} \quad \text{on } \eta = 0, \quad |\xi| < 1, \tag{8}$$

where p is the pressure, defined by the scaling

$$\hat{p} = qU_0^2 p/Re. \tag{9}$$

We expand the solution to (2) and (4) as an asymptotic series in $\delta\, Re^{\frac{1}{4}}$,

$$\psi_+ = \psi_{+0} + O(\delta\, Re^{+\frac{1}{4}}) \tag{10a}$$

and

$$\psi_- = \psi_{-0} + O(\delta\, Re^{+\frac{1}{4}}), \tag{10b}$$

in which case

$$\nabla^4\psi_{+0} = \nabla^4\psi_{-0} \equiv 0. \tag{11}$$

Thus to leading order the solution in the basement satisfies the Stokes equations of motion. The asymptotic expansion (10) breaks down when

$$\delta Re^{\frac{1}{4}}\eta = O(1) \tag{12}$$

and that region forms the lower deck. The complete statement of the problem for ψ_{+0} and ψ_{-0} also requires

$$\psi_{+0} \to 0 \quad \text{as} \quad \xi^2+\eta^2 \to \infty, \tag{13a}$$

$$\psi_{-0} \to 0 \quad \text{as} \quad \eta \to -\infty, \quad |\xi| < 1, \tag{13b}$$

together with the no-slip conditions on the walls

$$\psi_{+0\xi} = \psi_{-0\eta} \equiv 0 \quad \text{on} \quad \eta = 0, \quad |\xi| \geqslant 1, \tag{14a}$$

and

$$\psi_{-0\xi} = \psi_{-0\eta} \equiv 0 \quad \text{on} \quad \eta < 0, \quad |\xi| = 1. \tag{14b}$$

We define the velocities at the mouth of the slot to be

$$U(\xi) = \psi_{+0\eta}|_{\eta=0}, \quad |\xi| \leqslant 1, \tag{15a}$$

and

$$V(\xi) = -\psi_{+0\xi}|_{\eta=0}, \quad |\xi| \leqslant 1. \tag{15b}$$

The upper basement region ($\eta > 0$)

The solution for ψ_{+0} is given by Sneddon (1951, p. 296):

$$\psi_{+0} = \frac{1}{\pi}\int_{-1}^{1}\frac{\eta^2\,U(\xi')\,d\xi'}{\eta^2+(\xi-\xi')^2} - \frac{1}{\pi}\int_{-1}^{1} V(\xi')\left[\tan^{-1}\left(\frac{\xi-\xi'}{\eta}\right) + \frac{\eta(\xi-\xi')}{\eta^2+(\xi-\xi')^2}\right]d\xi'. \tag{16}$$

From this solution the pressure at the mouth of the slot is

$$p_+|_{\eta=0} = -\frac{2q}{\pi}\int_{-1}^{1}\frac{V(\xi')\,d\xi'}{(\xi-\xi')^2}, \tag{17}$$

whilst the shear on the wall and at the mouth of the slot is

$$\sigma_+|_{\eta=0} = U_1 + \frac{2q}{\pi}\int_{-1}^{1}\frac{U(\xi')\,d\xi'}{(\xi-\xi')^2}, \tag{18}$$

where $U_1 = \mathscr{U}'(0)$.

In Stokes-flow problems with sources and sinks it is conventional to take V to behave like a delta function and to assume that U vanishes. The above results show that even in the case of a point singularity it is incorrect to assume such a boundary condition unless

$$K = \int_{-1}^{1} U(\xi')\,d\xi' \equiv 0.$$

Indeed far from the hole

$$p_+|_{\eta=0} \approx -2/\pi\xi^2 + O(\xi^{-3}), \tag{19}$$

since by construction

$$\int_{-1}^{1} V(\xi')\,d\xi' = 1,$$

and

$$\sigma_+|_{\eta=0} \approx U_1 + 2Kq/\pi\xi^2 + O(\xi^{-3}). \tag{20}$$

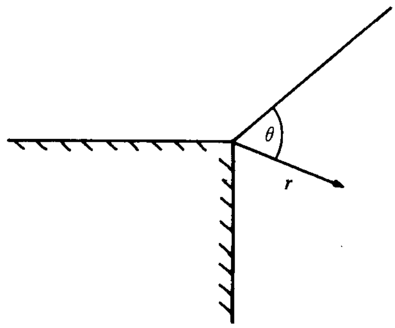

FIGURE 3. Geometry of 270° corner.

Flow near the corners

The expressions (17) and (18) for the pressure and the shear in general have a singularity near $|\xi| = 1$. Consider

$$I(x) = \int_{-1}^{1} \frac{f(\xi)}{(x-\xi)^2} d\xi, \tag{21}$$

where $f(\xi) = 0$ for $|\xi| = 1$. Then near $x = 1$

$$I(1+\epsilon_1) = \int_{\epsilon_1}^{2+\epsilon_1} \frac{f(1+\epsilon_1-t)}{t^2} dt$$

and unless $f'(1) = 0$ there will be a singularity as $\epsilon_1 \to 0$. If $f(1+\epsilon_1) \sim \epsilon_1^\lambda$ when $0 < \epsilon_1 \ll 1$ with $\lambda < 1$, the singularity will be like $\epsilon_1^{\lambda-1}$.

Flow near a corner has been studied by Dean & Montagnon (1949), Moffatt (1964) and Weinbaum (1968). We illustrate the geometry of a right-angled obtuse corner in figure 3. The solution for the stream function can be found as a series

$$\psi = \sum_{n=1}^{\infty} r^{\lambda_n} g_n(\theta), \tag{22}$$

where (r, θ) are polar co-ordinates illustrated in figure 3. For small r the solution is found to be approximately

$$\psi \approx r^\lambda f_k(\theta) + r^\mu g_\mu(\theta) + o(r^\lambda + r^\mu), \tag{23}$$

where $f_\lambda(\theta)$ is an even function of θ and $g_\mu(\theta)$ is odd. Weinbaum shows that $\lambda = 1{\cdot}544$ and $\mu = 1{\cdot}909$. The functions f_λ and g_μ are

$$f_\lambda(\theta) = A\left[\cos\lambda\theta - \frac{\cos\lambda\alpha}{\cos(\lambda-2)\alpha}\cos(\lambda-2)\theta\right] \tag{24a}$$

and

$$g_\mu(\theta) = B\left[\sin\mu\theta - \frac{\sin\mu\alpha}{\sin(\mu-2)\alpha}\sin(\mu-2)\theta\right], \tag{24b}$$

where $\alpha = \frac{3}{4}\pi$. From this we see that near the corners the shear and pressure will have singularities $O(r^{-0{\cdot}455})$ as $r \to 0$.

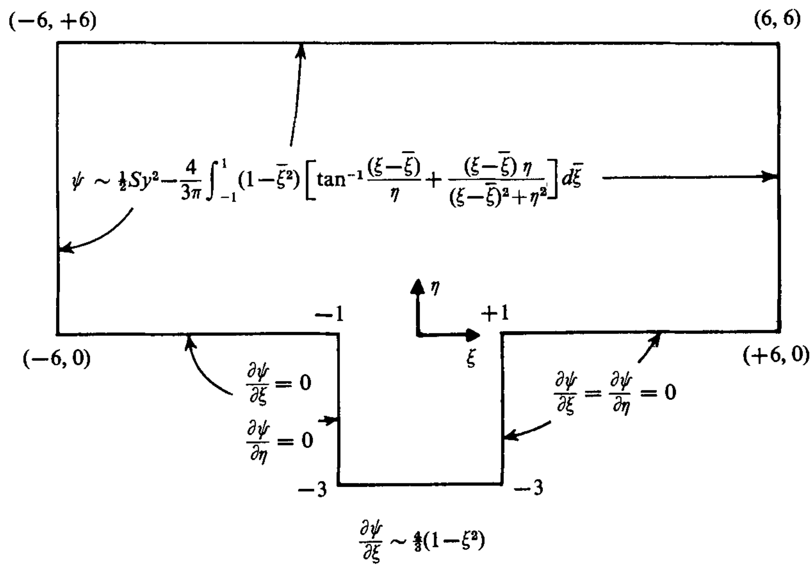

FIGURE 4. Region of finite-difference solution, showing boundary conditions at the walls and at the edge of the flow region.

A numerical solution for the basement region

In order to obtain a solution for the basement region we have solved the biharmonic equation using a finite-difference scheme. Since the main interest in the solution for the basement region is the computation of K,

$$K = \int_{-1}^{1} U(\xi)\, d\xi,$$

we have used a coarse uniform grid (spacing 0·2) in the regions $-6 \leqslant \xi \leqslant 6, 0 \leqslant \eta \leqslant 6$ and $-3 \leqslant \eta \leqslant 0, -1 \leqslant \xi \leqslant 1$. The region of solution is shown in figure 4. The boundary conditions which involve derivatives at the wall have been simulated by introducing fictitious grid points outside the flow region. On $\xi = \pm 6,\ 0 \leqslant \eta \leqslant 6$ and on $\eta = 6$, $-6 \leqslant \xi \leqslant 6$ the flow field was simulated by introducing grid points outside the flow region and then calculating the stream function from the formula

$$\psi \approx \tfrac{1}{2}Sy^2 - \frac{4}{3\pi}\int_{-1}^{1}(1-\xi'^2)\left\{\tan^{-1}\left(\frac{\xi-\xi'}{\eta}\right) + \frac{(\xi-\xi')\eta}{(\xi-\xi')^2+\eta^2}\right\} d\xi'. \tag{25}$$

Equation (25) is the form of ψ obtained by assuming plane Poisseuille flow at the mouth of the slit. Clearly this will be in error because of the deviation of the normal velocities at the slot mouth from Poiseuille flow and because of the existence of transverse velocities at the slot mouth. However for large S the stream function at the boundary of the computational area is dominated by the linear shear term, whilst for small S, as we show below, the corrections are proportional to S and hence asymptotically small compared with the linear shear term as one moves far from the slot. The finite-difference equations were solved using an over-relaxation method with iterations in alternate directions. We note that the flow through the side slot has been scaled to have unit flux, which makes the upstream shear $S = U_1/q$. Rewriting (1) and (3) we have, if $\eta > 0$,

$$\hat{u} = U_0 q\delta[S\eta + \partial\psi_{0+}/\partial\eta]$$

and $$\hat{v} = U_0 q\delta[-\partial\psi_0/\partial\xi].$$

Thus we may scale q out of the problem and have only S as an independent parameter. If ψ_{ij} is the value of the stream function at the point (i,j) then the new value ψ'_{ij} at each iteration is

$$\begin{aligned}\psi'_{ij} = \alpha[8(&\psi_{i+1,j}+\psi_{i-1,j}+\psi_{i,j+1}+\psi_{i,j-1}) \\ &-2(\psi_{i+1,j+1}+\psi_{i-1,j+1}+\psi_{i+1,j-1}+\psi_{i-1,j-1}) \\ &-(\psi_{i+2,j}+\psi_{i-2,j}+\psi_{i,j+2}+\psi_{i,j-2})]+(1-\alpha)\psi_{ij}. \end{aligned} \quad (26)$$

After some experimentation we chose $\alpha = 1\cdot9$ as a value which gave sufficiently rapid convergence. However for $S = 2$ convergence was improved by taking $\alpha = 1\cdot5$. The total computational grid consisted of 1982 grid points. The solution was continued until

$$\sum_{i,j}' |\psi'_{ij}-\psi_{ij}|$$

was less than some predetermined value E. Usually E was chosen as 0·05 but for some runs the accuracy of the solution was checked by taking $E = 0\cdot025$. The values of K were found by extrapolation assuming an exponential approach to the true value. This was suggested by Hinch (1976, private communication). Thus if after N iterations K_N is the calculated value of K, we compute K by assuming

$$K_N = K\,e^{-\lambda/N}$$

for some value of λ. The extrapolated value for K was generally within 10 % of the value at the last iteration.

We present in figures 5 (a)–(d) the computed contours of the flow for $S = 0\cdot05$, $0\cdot25$, $0\cdot5$ and $1\cdot0$. It can be seen that for $S \leqslant 0\cdot5$ there is a stagnation point on the plate away from the downstream corner. Thus in the immediate vicinity of the corner there may be a region of reversed flow. On a length scale comparable with the slot width the shear is generally increased. We show this in figure 6, where the shear on the plate is plotted for $U_1 = 1$ and $q = 0\cdot5$, $1\cdot0$ and $2\cdot0$. These graphs were computed from (18) as the grid used was too coarse to allow accurate computation of the shear on the wall. In figure 7 we plot the value of the parameter K/q as a function of S. We find that K/q is proportional to S, which means that the shear correction far from the hole depends only on U_1 and not on q. We calculate

$$K/q = 0\cdot41 U_1/q.$$

As far as the accuracy of our work is concerned, given the coarseness of the grid, the solution to the finite-difference equations is accurate to within a few per cent. However we have not determined the effect of increasing the resolution of the grid, or of increasing the computational area.

3. Analysis of the lower-deck region

The asymptotic series for ψ_+ given by (10) breaks down when the term $\delta^2 Re^{\frac{1}{2}}\eta\nabla^2\psi_{+\xi}$ becomes comparable with the term $\nabla^4\psi_+$ in (2). As one moves away from the slot into the lower region of the boundary layer, inertia effects become important and simultaneously the slot takes on the appearance of a point disturbance, but with novel

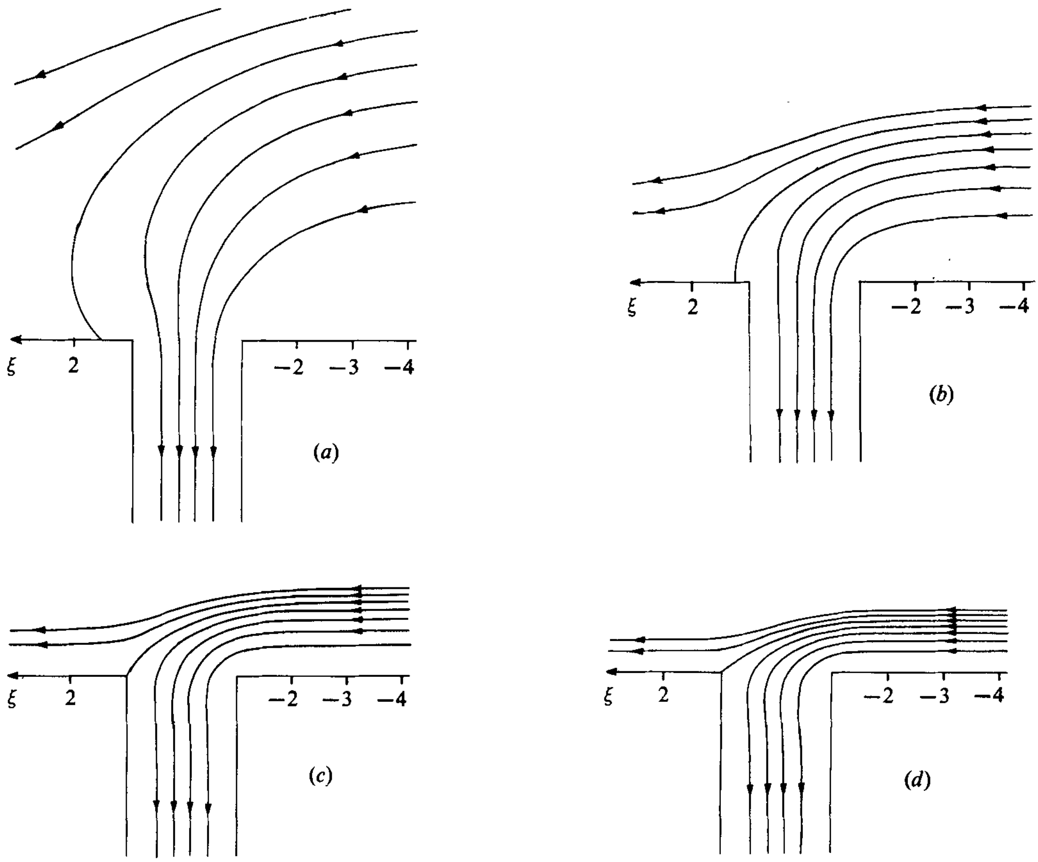

FIGURE 5. Streamlines of basement solution for (*a*) $S = 0{\cdot}05$, (*b*) $S = 0{\cdot}25$, (*c*) $S = 0{\cdot}5$ and (*d*) $S = 1{\cdot}0$.

boundary conditions, namely that there is a delta-function disturbance in both the normal and the transverse direction.

Let

$$\epsilon = \delta\, Re^{\frac{1}{4}} \ll 1,$$

and define (x, y) by

$$x = \epsilon\xi, \quad y = \epsilon\eta. \tag{27}$$

This corresponds to $\hat{x} = Re^{-\frac{3}{4}}xL$ and $\hat{y} = Re^{-\frac{3}{4}}yL$. The velocities are scaled by

$$\hat{u} = U_0[\mathscr{U}(Re^{-\frac{1}{4}}y) + q\delta\epsilon\, \partial\psi/\partial y], \tag{28a}$$

$$\hat{v} = -U_0 q\delta\epsilon\, \partial\psi/\partial x, \tag{28b}$$

where the stream function in the lower deck is denoted by ψ to distinguish it from the basement stream function ψ_+. The governing equation for ψ is then

$$\nabla^4\psi + \frac{\delta\, Re^{-\frac{1}{4}}}{q}\mathscr{U}'''(Re^{-\frac{1}{4}}y) = q\delta^2\, Re^{\frac{1}{2}}\frac{\partial(\nabla^2\psi, \psi)}{\partial(x, y)} + Re^{\frac{1}{4}}[\mathscr{U}(Re^{-\frac{1}{4}}y)\,\nabla^2\psi_x - \delta^2\mathscr{U}''(Re^{-\frac{1}{4}}y)\,\psi_x]. \tag{29}$$

The boundary conditions are

$$\psi_y|_{y=0} = \epsilon^{-1}U(\epsilon^{-1}x), \quad \psi_x|_{y=0} = \epsilon^{-1}V(\epsilon^{-1}x) \tag{30a, b}$$

and

$$\psi \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty. \tag{30c}$$

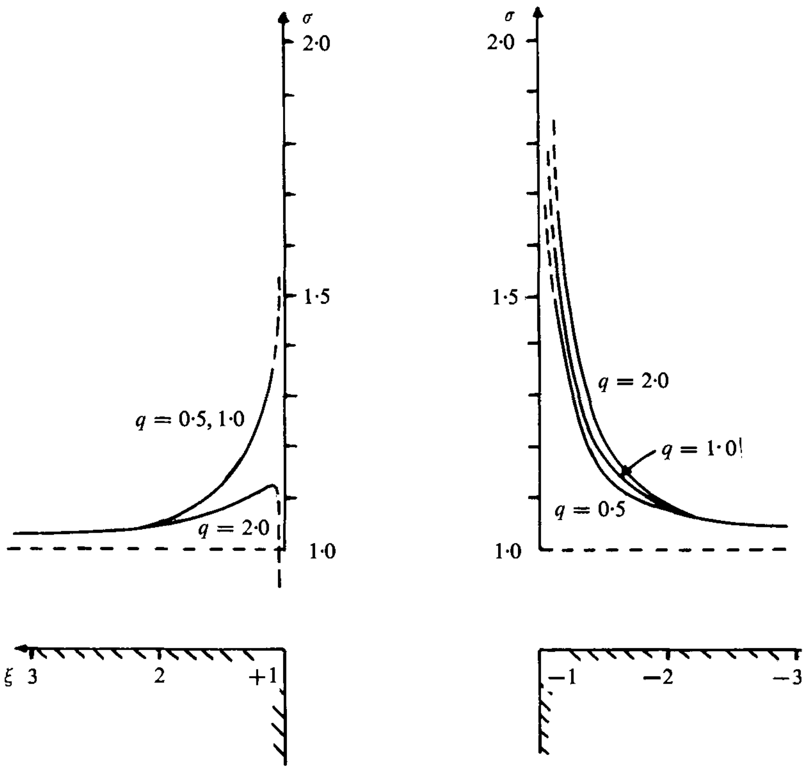

FIGURE 6. Calculated shear at the wall for $U_1 = 1{\cdot}0$ and $q = 0{\cdot}5$, $1{\cdot}0$ and $2{\cdot}0$.

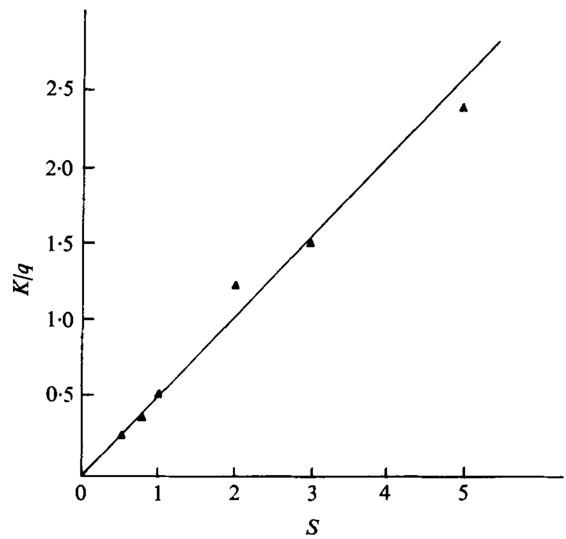

FIGURE 7. Values of K/q plotted against S. ▲, computed values.

Let $\bar{f}(k)$ be the Fourier transform of a function $f(x)$, defined by

$$\bar{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx. \tag{31}$$

Taking Fourier transforms of the boundary conditions gives

$$\bar{\psi}_y|_{y=0} = \bar{U}(\epsilon k), \quad \bar{\psi}|_{y=0} = -\bar{V}(\epsilon k)/ik. \tag{32a,b}$$

We now expand $\bar{\psi}$ as an asymptotic series

$$\bar{\psi} = \bar{\psi}_0 + O(Re^{-\frac{1}{4}}) \tag{33}$$

to obtain

$$D^4\bar{\psi}_0 - ikU_{1y} D^2\bar{\psi}_0 = 0, \tag{34}$$

where

$$D^2 = d^2/dy^2 - k^2, \tag{35}$$

together with the boundary conditions

$$\bar{\psi}_{0y}|_{y=0} = K, \quad \bar{\psi}|_{y=0} = -1/ik \tag{36a,b}$$

and

$$\bar{\psi} \to 0 \quad \text{as} \quad y \to \infty. \tag{36c}$$

The expansion of the boundary conditions breaks down if $k^{-1} = O(\epsilon^{-1})$, i.e. $x = O(\epsilon)$, where one would be back in the basement and the side slot would no longer have the appearance of a point singularity.

Let

$$\bar{G} = D^2\bar{\psi}_0; \tag{37}$$

then

$$(D^2 - ikU_1 y)\bar{G} = 0. \tag{38}$$

This is a variant of Airy's equation and appeared in Stewartson (1968), where the boundary conditions were more complicated and a Weiner–Hopf technique was used to determine the solution for boundary-layer flow near the trailing edge of a flat plate.

In the literature dealing with the application of triple-deck analyses to humps (Smith 1973; Hunt 1971; Brighton & Jackson 1978), it is generally assumed that variations in the x direction are small compared with variations in the y direction, and hence the operator D becomes

$$D^2 \approx d^2/dy^2. \tag{39}$$

Indeed, from Stewartson & Williams (1969) and Smith (1973), when the scale in the x direction is not longer than that in the y direction the solution no longer has a true triple-deck structure as the pressure will vary across the boundary layer. The case here is mathematically similar to that of a short hump of dimensions $Re^{-\frac{3}{4}}L \times Re^{-\frac{3}{4}}L$.

Let $\zeta(y,k) = (ik)^{\frac{1}{3}}(U_1^{\frac{1}{3}} y - ikU_1^{-\frac{2}{3}})$, where $(ik)^{\frac{1}{3}} = (0+ik)^{\frac{1}{3}}$; then substituting (39) into (38) we have Airy's equation for $\bar{G}(\zeta)$:

$$d^2\bar{G}/d\zeta^2 - \zeta\bar{G} = 0.$$

We eliminate solutions of the form $\mathrm{Bi}(\zeta)$ to satisfy the boundary condition (36) and take

$$\bar{G} = \alpha(k)\,\mathrm{Ai}(\zeta)' \tag{40}$$

where $\alpha(k)$ is an unknown function of k.

To obtain solutions of (37) we take

$$\bar{\psi}_0 = A(k)\, e^{-|k|y} + \frac{\alpha(k)}{|k|}\int_y^{\infty} \mathrm{Ai}(\zeta')\sinh|k|(y-y')\, dy', \tag{41}$$

where $\zeta' = \zeta(y', k)$. The boundary conditions (36) then give

$$\alpha(k) = (K + i\,\mathrm{sgn}\,k)/\mathrm{Ci}^*(k), \tag{42}$$

where $$\mathrm{Ci}^*(k) = \int_0^\infty \mathrm{Ai}(\zeta')\,e^{-|k|y'}\,dy'.$$

The function $\mathrm{Ci}^*(k)$ can be related to the function $\mathrm{Ci}(\omega)$ defined by Stewartson (1968). Let $y' = U_1^{-\frac{1}{3}} Y$ and $k = U_1^{\frac{1}{3}}\omega$; then

$$\mathrm{Ci}^*(k) = U_1^{-\frac{1}{3}}\,\mathrm{Ci}(\omega).$$

Thus we may rewrite (42) as

$$\alpha(k) = (K + i\,\mathrm{sgn}\,k)/U_1^{-\frac{1}{3}}\,\mathrm{Ci}(U_1^{-\frac{1}{3}}k), \tag{43}$$

where $$\mathrm{Ci}(\omega) = \int_0^\infty \mathrm{Ai}((i\omega)^{\frac{1}{3}}(Y - i\omega))\,e^{-|\omega|Y}\,dY. \tag{44}$$

We also have

$$A(k) = -\frac{1}{ik} + \frac{\alpha}{|k|}\int_0^\infty \mathrm{Ai}(\zeta')\sinh|k|\,y'\,dy'. \tag{45}$$

To calculate the shear on the wall we must evaluate $\bar{\psi}_{0yy}|_{y=0}$. We have

$$\bar{\psi}_{0yy}|_{y=0} = ik + \frac{K}{U_1^{-\frac{1}{3}}} + \frac{i\,\mathrm{sgn}\,k}{\mathrm{Ci}(U_1^{-\frac{1}{3}}k)}\mathrm{Ai}(\theta), \tag{46}$$

where $$\mathrm{Ai}(\theta) = \mathrm{Ai}(-(ik)^{\frac{1}{3}}ikU_1^{-\frac{4}{3}}),$$

and thus $$\bar{\psi}_{0yy}|_{y=0} = ik + (K + i\,\mathrm{sgn}\,k)/U_1^{-\frac{1}{3}}K(U_1^{-\frac{1}{3}}k), \tag{47}$$

where $K(\omega)$ was defined by Stewartson (1968) to be

$$K(\omega) = \mathrm{Ci}(\omega)/\mathrm{Ai}(-i\omega(i\omega)^{\frac{1}{3}}). \tag{48}$$

We use the asymptotic expansions of $K(\omega)$ for $|\omega| \gg 1$ and $|\omega| \ll 1$ obtained by Stewartson to determine the asymptotic forms of the shear on the wall for $|x| \ll 1$ and $|x| \gg 1$. However because of the restriction $k^{-1} = O(\epsilon^{-1})$ the lower-deck solution does not allow the shear to be obtained very near the slit. The pressure $p_0(x)$ on the wall is found from

$$\bar{p}_0(k) = (ik)^{-1}D^2\bar{\psi}_0|_{y=0}, \tag{49}$$

whence $$\bar{p}_0(k) = \alpha(k)\,(ik)^{-\frac{2}{3}}U_1^{\frac{1}{3}}\,\mathrm{Ai}'(\theta). \tag{50}$$

Asymptotic analysis of the lower deck

Here we shall derive the leading terms of the asymptotic series for the shear and the pressure on the wall when $|x| \gg 1$ and $|x| \ll 1$. Stewartson (1968) has shown that for $|\omega| \gg 1$

$$K(\omega) \approx 1/2\,|\omega|, \tag{51}$$

whilst for $|\omega| \ll 1$ $$\mathrm{Ci}(\omega) \approx 1/3(i\omega)^{\frac{1}{3}}. \tag{52}$$

The shear on the wall, under the scaling (28), is given by

$$\sigma_+|_{y=0} = U_1 + q\delta^2\,\mathrm{Re}^{\frac{1}{2}}\,\psi_{0yy}|_{y=0}, \tag{53}$$

and for $|k| \approx 1$ and $\epsilon|k| \ll 1$,

$$\bar{\psi}_{0yy}|_{y=0} \approx ik + 2(K + i\,\mathrm{sgn}\,k)\,|k|. \tag{54}$$

Then using Lighthill (1958) to invert the Fourier transforms gives

$$\sigma_+|_{y=0} \approx U_1 + 2K\delta^2 \, Re^{\frac{1}{2}}/\pi x^2, \quad \delta \, Re^{\frac{1}{4}} \ll x \ll 1. \tag{55}$$

This agrees exactly with the expansion of (20), the basement solution for large $|\xi|$.

If $|k| \ll 1$ we have

$$1/K(\omega) \approx 3 \, \mathrm{Ai}(0) \, (i\omega)^{\frac{1}{3}}, \tag{56}$$

and hence

$$\bar{\psi}_{0yy}|_{y=0} \approx ik + 3(K + i \, \mathrm{sgn} \, k) \, \mathrm{Ai}(0) \, (ikU_1^{-\frac{1}{2}})^{\frac{1}{3}}/U_1^{-\frac{1}{2}}. \tag{57}$$

Thus inverting the Fourier transforms yields

$$\sigma_+|_{y=0} \approx U_1 + \frac{3q\delta^2 \, Re^{\frac{1}{2}} \, \mathrm{Ai}(0) \, \Gamma(\frac{1}{3}) \, U_1^{\frac{1}{3}}}{\pi |x|^{\frac{4}{3}}} \left\{ K \cos\frac{\pi}{6}(1 - 4 \, \mathrm{sgn} \, x) - \sin\frac{\pi}{6}(1 - 4 \, \mathrm{sgn} \, x) \right\}. \tag{58}$$

It is of some interest to examine the components of the shear in dimensional form. If we let $\hat{K}$ be the dimensional equivalent of

$$\int_{-1}^{1} U(\xi) \, d\xi,$$

then

$$\hat{K} = \delta a \, U_0 q K.$$

For $|\hat{x}/L| = O(\delta)$

$$\left.\frac{\partial \hat{u}}{\partial \hat{y}}\right|_{\hat{y}=0} = \frac{U_0}{L}\left[U_1 + \frac{2\hat{\kappa}}{U_0 L\pi}\left|\frac{\hat{x}}{L}\right|^{-2}\right]. \tag{59}$$

If $|\hat{x}/L| = O(Re^{-\frac{3}{4}})$, which here implies, since $\delta \, Re^{\frac{1}{4}} \ll 1$, that $\delta \ll Re^{-\frac{1}{4}} \ll 1$,

$$\left.\frac{\partial \hat{u}}{\partial \hat{y}}\right|_{\hat{y}=0} = \frac{U_0}{L}\left\{U_1 + \frac{3\hat{\kappa} \, Re^{\frac{1}{6}} \, U_1^{\frac{1}{3}}}{U_0 L\pi} \mathrm{Ai}(0) \, \Gamma(\tfrac{1}{3}) \cos\frac{\pi}{6}\left(1 - 4 \, \mathrm{sgn}\left(\frac{\hat{x}}{L}\right)\right)\left|\frac{\hat{x}}{L}\right|^{-\frac{4}{3}}\right.$$
$$\left. - \frac{3\hat{Q} \, Re^{\frac{2}{3}} \, U_1^{\frac{1}{3}}}{U_0 L\pi} \mathrm{Ai}(0) \, \Gamma(\tfrac{1}{3}) \sin\frac{\pi}{6}\left(1 - 4 \, \mathrm{sgn}\left(\frac{\hat{x}}{L}\right)\right)\left|\frac{\hat{x}}{L}\right|^{-\frac{4}{3}}\right\}. \tag{60}$$

We see that in this region of the wall the shear is strongly dependent on the source strength, with a multiplier of $Re^{\frac{2}{3}}$ whilst the transverse impulse has a multiplier of $Re^{\frac{1}{6}}$. Thus, far from the slot, the source-like behaviour dominates, showing that in this region the approximation of zero transverse velocities at the mouth of the slot is a good approximation.

If $|k| \ll 1$

$$\bar{p}_0 \approx U_1^{\frac{2}{3}}(K - i \, \mathrm{sgn} \, k) \, 3(ik)^{-\frac{1}{3}} \, \mathrm{Ai}'(0). \tag{61}$$

Taking transforms, we find

$$p_0(x) \approx \frac{3U_1^{\frac{2}{3}} \, \mathrm{Ai}'(0) \, \Gamma(-\frac{1}{3})}{\pi |x|^{\frac{2}{3}}}\left[K \cos\frac{\pi}{6}(1 + 2 \, \mathrm{sgn} \, x) - \sin\frac{\pi}{6}(1 + 2 \, \mathrm{sgn} \, x)\right]. \tag{62}$$

The decay, like $|x|^{-\frac{2}{3}}$, agrees with that obtained by Brighton & Jackson (1978) for the decay of the wall pressure on a plate with a long slender hump on it.

When $|k| \gg 1$

$$\bar{p}_0 = U_1^{\frac{5}{6}}(K - i \, \mathrm{sgn} \, k) \, (ik)^{\frac{1}{3}} \, \mathrm{Ai}'(\theta)/ik \, K(kU_1^{-\frac{1}{2}}) \, \mathrm{Ai}(\theta). \tag{63}$$

Since $|\arg \theta| = \frac{1}{3}\pi$, we may use the expansions for $\mathrm{Ai}(\theta)$ and $\mathrm{Ai}'(\theta)$ for large $|\theta|$ given by Abramowitz & Stegun (1965, p. 44) to obtain

$$\mathrm{Ai}'(\theta)/\mathrm{Ai}(\theta) \approx -\theta^{\frac{1}{2}}. \tag{64}$$

Hence
$$\bar{p}_0 \approx 2(K - i\,\mathrm{sgn}\,k)\,ik, \tag{65}$$
and thus
$$p_0(x) \approx 2/\pi x^2, \tag{66}$$
which agrees with the expansion of the basement solution for large $|\xi|$. In dimensional terms, if $|\hat{x}/L| = O(\delta)$
$$\hat{p} \approx (\hat{Q}/U_0 L)\, U_0^2\, Re^{-\frac{1}{2}}\, |\hat{x}/L|^{-2}, \tag{67}$$
and if $|\hat{x}/L| = O(Re^{-\frac{3}{4}})$ we have
$$\hat{p} \approx \frac{3U_0^2 U_1^{\frac{2}{3}} \mathrm{Ai}'(0)\,\Gamma(-\frac{1}{3})}{\pi} \left|\frac{\hat{x}}{L}\right|^{-\frac{2}{3}} \left\{ \frac{\hat{K}}{U_0 L} Re^{-\frac{2}{3}} \cos\frac{\pi}{6}\left(1 + 2\,\mathrm{sgn}\left(\frac{\hat{x}}{L}\right)\right) - \frac{\hat{Q}}{U_0 L} Re^{-\frac{1}{6}} \sin\frac{\pi}{6}\left(1 + 2\,\mathrm{sgn}\left(\frac{\hat{x}}{L}\right)\right)\right\}. \tag{68}$$
Once again the dominant contribution will be from the source term $\hat{Q}\, Re^{-\frac{1}{6}}\, U_0^{-1} L^{-1}$.

4. Conclusions

We have seen how a consistent model of flow through a small side slot may be developed using a combination of modern boundary-layer theory and classical low Reynolds number flow theory. From a practical viewpoint, the model is severe in its assumptions of small flux through the side slot and small dimensions of the slot relative to the boundary-layer thickness. We have excluded many of the interesting situations in which the main body of the boundary layer is either blown off the plate or is largely removed by the side slot. What we have shown is that adequate attention must be paid to the dynamics near and in the side slot if an accurate model is to be obtained.

In the descending aorta, if we suppose that a disturbance boundary layer develops on the walls, we may take $L = O(10\text{–}100\,\mathrm{mm})$ and $a = O(1\,\mathrm{mm})$. The Reynolds number of the boundary-layer flow will be $Re = RL/A$, where R is the Reynolds number based on the aortic radius A. If $A = O(10\,\mathrm{mm})$ then $R < Re < 10R$ and since $a = \delta\, Re^{-\frac{1}{2}} L$ we have $10^{-2}R < \delta < R$ and thus
$$10^{-2}R^{\frac{5}{4}} < \delta\, Re^{\frac{1}{4}} < 1{\cdot}78\, R^{\frac{5}{4}}.$$
In the descending aorta, where the peak values of the Reynolds number are $O(10^2\text{–}10^3)$, it is probable that $\delta > 1$ and $\delta\, Re^{\frac{1}{4}} > 1$ for much of the flow cycle, whilst $\delta < 1$ and $\delta\, Re^{\frac{1}{4}} < 1$ only when $R = O(10)$. Further work is necessary before this theory can be conclusively applied to the intercostal arteries. However we can note that (20) and (60) show that on the scale of the side slot the wall shear stress increases on both sides of the slot. Our numerical work shows that there may be a region of reversed shear near the downstream corner, in which case the shear on the upstream wall is larger than that on the downstream wall. Presumably, as well as faster-moving fluid being moved to the downstream wall, the fluid upstream is accelerated, leading to increased wall shear stress upstream of the slot.

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